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The unstable spectrum of swirling gas flows

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The asymptotic structure of the discrete spectrum of a compressible inviscid swirling flow with arbitrary radial distributions of density, pressure and velocity is described for disturbances with large wavenumbers. It is shown that discrete eigenmodes are unstable when a criterion derived by Eckhoff & Storesletten (1978) is satisfied. In general, these modes are characterized by a length scale of order $|m|^{-3/4}$ where $|m| \gg 1$ is the azimuthal wavenumber of the disturbance. They have a spatial structure similar to the incompressible modes obtained by Leibovich & Stewartson (1983). In the particular case of solid-body rotation with a positive gradient of entropy, the unstable discrete spectrum contains modes which scale with $|m|^{-1/2}$. If the modes are localized near a solid boundary, they scale with $|m|^{-2/3}$.

1. Introduction

In the present paper, we examine the temporal linear stability of swirling motions of an inviscid compressible flow in adiabatic evolution. Neglecting the effect of gravitational acceleration, the equilibrium (or base) flow is described in polar cylindrical coordinates by its radial distributions of velocity $V(r)\boldsymbol{e}_\theta + W(r)\boldsymbol{e}_z$, pressure $P(r)$, density $R(r)$ and entropy $S(r)$. Pressure and density are related through the dynamical relation $P'/R = V^2/r$ (the prime denotes differentiation with respect to r), and thermodynamic variables by the equation of state $P = P(R, S)$. The local speed of sound $C(r)$ of the equilibrium flow is defined by $C^2 = (\partial P/\partial R)_S$. For instance, in an ideal gas with constant ratio of specific heats γ, we have the usual relations: $S = \ln(P/R^\gamma)$ and $C^2 = \gamma P/R$.†

Various stability criteria have been derived in the literature for arbitrary radial distributions of velocity, pressure and density fields. A review may be found in Le Duc (2001). We focus our attention here on a criterion derived by Eckhoff & Storesletten (1978): they proved that the flow is unstable if for some $r > 0$

$$\frac{V^2}{r}\left(\frac{R'}{R} - \frac{V^2}{rC^2}\right)\left(W'^2 + \left(V' - \frac{V}{r}\right)^2\right) + \frac{2V}{r}\left(V' - \frac{V}{r}\right)\left(W'^2 + V'^2 - \frac{V^2}{r^2}\right) < 0. \quad (1.1)$$

This is a sufficient condition for instability.

Eckhoff (1984) pointed out that if $C^2 = \infty$ and $R' = 0$, (1.1) is analogous to the criterion derived by Leibovich & Stewartson (1983) for an incompressible swirling flow. However, the methods used respectively by Eckhoff & Storesletten (the local

† The explicit form of the equation of state can be avoided in the stability analysis (Howard 1973); all results presented here are formally valid for any gas in isentropic evolution.

theory) and by Leibovich & Stewartson (the modal approach) are different: the former is associated with the continuous spectrum, the latter with the discrete spectrum. The objective of the present paper is, in the case of swirling gas flows, to connect the two approaches. In particular, we will show that unstable discrete modes exist if (1.1) is satisfied.

After a brief review of relevant stability conditions (§ 2), the asymptotic structure of the instability modes in a compressible swirling flow is described in § 3 and § 4.

2. Modal approach versus local theory

2.1. *Sufficient conditions for stability*

In the modal approach (a terminology which we prefer to the usual 'normal' modes because the operators are generally non-normal), linear disturbances are sought as

$$[\boldsymbol{u}(r), \mathrm{i}p(r), \rho(r)]\mathrm{e}^{\mathrm{i}(m\theta+kz-\omega t)}, \tag{2.1}$$

where, in a temporal analysis, m is integer, k real and $\omega=\omega_R+\mathrm{i}\omega_I$ complex. If $\omega_I>0$ then the discrete spectrum contains unstable modes.

To shorten notation, we define $\Omega(r)=V/r$ the angular velocity,

$$\Delta(r) = m\Omega + kW - \omega \tag{2.2}$$

the complex Doppler shifted frequency, and $\Delta'(r)=m\Omega'+kW'$ its derivative. Also let

$$\Phi(r) = 2\Omega(2\Omega + r\Omega'), \quad N^2(r) = r\Omega^2(R'/R - r\Omega^2/C^2) \tag{2.3}$$

be respectively the Rayleigh discriminant of the flow and, in analogy with stratified flows, the square of the Brunt–Vaïssälä frequency (here N^2 positive or negative).

Lalas (1975) and Warren (1975) proved independently that, if for some (m, k) the flow is unstable ($\omega_I>0$), then there exists some radius r such that†

$$\omega_I^2 \leqslant K^2(r), \quad K^2(r) = \left(2m\Omega + \tfrac{1}{2}r\Delta'\right)^2/(m^2+k^2r^2) - N^2 - \Phi. \tag{2.4}$$

A similar result was derived by Howard & Gupta (1962) for an incompressible homogeneous fluid ($N^2=0$). The maximum over r of $K^2(r)$ provides an upper bound for the growth rate, and if $K^2 \leqslant 0$ everywhere, (2.4) implies that disturbances with wavenumbers (m, k) are stable ($\omega_I=0$). This is a sufficient condition for stability. For instance, the flow is stable to axisymmetric disturbances ($m=0$) if $\frac{1}{4}W'^2-N^2-\Phi\leqslant 0$ everywhere (Howard 1973). More generally, Lalas and Warren proved that the flow is stable to any linear disturbance if $\frac{1}{4}(W'^2+r^2\Omega'^2)-N^2\leqslant 0$ everywhere.

Important is the case where

$$\Delta'(r) = m\Omega' + kW' = 0 \tag{2.5}$$

everywhere, for which (2.4) leads to

$$\omega_I^2 \leqslant \Sigma^2(r), \quad \Sigma^2(r) = (2m\Omega)^2/(m^2+k^2r^2) - N^2 - \Phi. \tag{2.6}$$

Condition (2.5) is ensured when W is uniform and $m=0$. In that case (2.6) shows that the flow is stable to axisymmetric disturbances if

$$N^2 + \Phi \geqslant 0 \tag{2.7}$$

† The prefactor $\frac{1}{4}$ on the right-hand side of inequality (3.7) in Lalas (1975) is erroneous.

everywhere. When Ω is uniform, the flow is stable to planar disturbances ($k=0$) if

$$N^2 \geqslant 0 \tag{2.8}$$

everywhere. When both Ω and W are uniform, then (2.5) is satisfied for any (m, k), and the corresponding disturbances are stable if $N^2+(2\Omega kr)^2/(m^2+k^2r^2) \geqslant 0$ everywhere. Therefore, the unstable discrete spectrum is empty if $N^2 \geqslant 0$ everywhere.

Finally, if Ω and W are not uniform and if there exists some r where (2.4) and (2.5) hold, we can conclude that instability implies:

$$\omega_I^2 \leqslant \Xi^2(r), \quad \Xi^2(r) = (2\Omega W')^2/(r^2\Omega'^2 + W'^2) - N^2 - \Phi. \tag{2.9}$$

Therefore if $\Xi^2 \leqslant 0$ everywhere, there exists no unstable mode characterized by (2.5).

2.2. *Sufficient conditions for instability*

Using a theory developed by Eckhoff (1981) and generalized by Lifschitz & Hameiri (1991), Eckhoff & Storesletten (1978) derived sufficient conditions for instability for perfect gas in swirling motions. Their method consists in looking for the response of the flow to an initially localized disturbance. The disturbance is approximated by a WKB expansion with leading term

$$[\boldsymbol{u}(\boldsymbol{x}, t), \varepsilon p(\boldsymbol{x}, t), \rho(\boldsymbol{x}, t)]\mathrm{e}^{\mathrm{i}\phi(\boldsymbol{x},t)/\varepsilon}, \quad 0 < \varepsilon \ll 1. \tag{2.10}$$

Defining $(\xi_\theta, \xi_z) = (\partial_\theta\phi, \partial_z\phi)$, Eckhoff & Storesletten demonstrated that the amplitudes of (2.10) grow exponentially if, for some r, the two following conditions are satisfied (details may be found in Le Duc 2001):

$$\xi_\theta\Omega' + \xi_z W' = 0, \tag{2.11}$$

$$(2\xi_\theta\Omega)^2/\left(\xi_\theta^2 + \xi_z^2 r^2\right) - N^2 - \Phi > 0. \tag{2.12}$$

Various cases may be identified. If W is uniform and $\Omega' \neq 0$, then (2.11) is satisfied only for axisymmetric short-wave disturbances ($\xi_\theta = 0$) and the flow is unstable if

$$N^2 + \Phi < 0 \tag{2.13}$$

somewhere (compressible Rayleigh's criterion). If Ω is uniform and $W' \neq 0$, then (2.12) with $\xi_z = 0$ shows that instability arises whenever

$$N^2 < 0 \tag{2.14}$$

somewhere. Finally, if Ω and W are both uniform, then relation (2.11) is always satisfied so that the flow is unstable if (2.12) holds somewhere, that is if for some r

$$N^2 + (2\Omega\xi_z r)^2/\left(\xi_\theta^2 + \xi_z^2 r^2\right) < 0. \tag{2.15}$$

The flow is therefore unstable when $N^2 < 0$ somewhere.

Finally, if $\Omega' \neq 0$ and $W' \neq 0$ in the flow, then (2.11) may be always satisfied and provides a relation between ξ_θ and ξ_z. Replacing (2.11) in (2.12), the flow is unstable if for some r, $\Xi^2 > 0$, where $\Xi^2(r)$ is defined in (2.9). This is equivalent to (1.1). Other criteria were derived by Eckhoff & Storesletten leading to algebraic growth. These weaker instabilities are outside the scope of the present paper.

2.3. *Discrete and continuous spectra*

As already mentioned, Eckhoff (1984) pointed out that (1.1) corresponds in the incompressible limit to the criterion derived by Leibovich & Stewartson (1983). However, the local theory and the modal approach used respectively by Eckhoff &

Storesletten and by Leibovich & Stewartson are quite different in nature. Indeed, although powerful for deriving stability criteria, the local theory has its own limitations: (i) it is restricted to short-wave disturbances; (ii) it gives no information about the modal structure of the instability; (iii) the actual disturbance approximated by (2.10) grows initially but may die out after a finite time. These restrictions are essentially due to the fact that the exponential growth rate of the disturbance amplitude is related to a point on the continuous (essential) spectrum (see Lebovitz & Lifschitz 1992).

On the contrary, the modal approach yields information on the discrete spectrum, for which unstable modes grow exponentially in the linear regime forever. Furthermore, the structure of the eigenmodes can generally be computed. It provides useful information for comparison with experiments or numerical results. However, the mathematical analysis related to the modal approach is much more difficult and generally restricted to flows with particular symmetries, such as swirling flows discussed here.

In our previous work (Le Duc & Leblanc 1999), following a procedure outlined by Bayly (1988), we showed how to construct discrete axisymmetric eigenmodes associated with the compressible analogue of Rayleigh's criterion for centrifugal instability (2.13). But this work was restricted to the particular case W uniform and $m=0$. In a recent paper, Sipp *et al.* (2005) described the asymptotic structure of modes localized in the core of a vortex characterized by a negative gradient of density, but their analysis was restricted to two-dimensional disturbances of a non-homogeneous incompressible flow. In the rest of the paper, we extend these asymptotic analyses to arbitrary radial distribution of density, pressure and velocity.

3. Asymptotic structure of the disturbances: the general case

3.1. *The eigenvalue problem*

With (2.1), the linearized equations may be written as (see for instance Warren 1975)

$$\frac{\mathrm{d}(ru)}{\mathrm{d}r} - \frac{\Delta'}{\Delta}(ru) - \left(\frac{2m\Omega}{r\Delta} - \frac{r\Omega^2}{C^2}\right)(ru) = \frac{r\Delta}{R}\left(\frac{1}{C^2} - \frac{m^2+k^2r^2}{r^2\Delta^2}\right)p,$$
$$\frac{\mathrm{d}p}{\mathrm{d}r} + \left(\frac{2m\Omega}{r\Delta} - \frac{r\Omega^2}{C^2}\right)p = \frac{R}{r\Delta}(N^2+\Phi-\Delta^2)(ru),$$

where $u(r)$ and $p(r)$ are the radial velocity and pressure disturbances. Equations corresponding to an incompressible flow are recovered with $C^2=\infty$, with either homogeneous density if $R'=0$ or inhomogeneous density if not.

With $\varphi = ru/\Delta$, we obtain $\varphi' - a\varphi = bp$ and $p' + ap = c\varphi$ where

$$a = \left(\frac{2m\Omega}{r\Delta} - \frac{r\Omega^2}{C^2}\right), \quad b = \frac{r}{R}\left(\frac{1}{C^2} - \frac{m^2+k^2r^2}{r^2\Delta^2}\right), \quad c = \frac{R}{r}(N^2+\Phi-\Delta^2).$$

By differentiation, the following second-order equation may be deduced:

$$\varphi'' - (\ln b)'\varphi' + (a(\ln b)' - a' - a^2 - bc)\varphi = 0.$$

Introducing $\psi = \varphi/b^{1/2}$, we finally obtain

$$\psi'' + Q(r;\omega)\psi = 0, \tag{3.1}$$
$$Q(r;\omega) = \tfrac{1}{2}(\ln b)'' - \tfrac{1}{4}(\ln b)'^2 + a(\ln b)' - a' - a^2 - bc. \tag{3.2}$$

Boundary conditions for $\psi(r)$ are the same as for radial velocity $u(r)$, that is $\psi = 0$ on the boundaries or at infinity. Possible singularities at the origin are avoided because the instabilities presented here are localized at non-zero radii (condition (1.1) cannot be satisfied at $r = 0$). Equation (3.1) forms an eigenvalue problem. It consists in finding, for fixed (k, m), the complex values of ω (the eigenvalues) for which smooth non-trivial solutions $\psi(r)$ exist (the eigenfunctions).

We are interested in the behaviour of solutions with large wavenumbers: $|m| \gg 1$ and/or $|k| \gg 1$. It is thus convenient to introduce a small parameter ε such that $0 < \varepsilon \ll 1$, and to define the following rescaled wavenumbers:

$$\tilde{m} = \varepsilon m, \quad \tilde{k} = \varepsilon k. \tag{3.3}$$

Both $\tilde{m}$ and $\tilde{k}$ are assumed of order unity; they may be chosen null separately in order to consider axisymmetric ($\tilde{m} = 0$) or planar ($\tilde{k} = 0$) disturbances. It is also convenient to define the real-valued function

$$\Lambda(r) = \tilde{m}\Omega + \tilde{k}W.$$

By construction, $\Lambda(r) = O(1)$, and from definition (2.2), $\Delta(r) = \varepsilon^{-1}\Lambda(r) - \omega$.

Each term of (3.2) may be expanded in powers of ε. At leading order, we obtain

$$Q(r;\omega) = -\frac{\tilde{m}^2 + \tilde{k}^2 r^2}{\varepsilon^2 r^2}\left(\frac{\Sigma^2}{\Delta^2} + 1\right) + O(\varepsilon^{-1}), \tag{3.4}$$

where $\Sigma^2(r)$ is a term of order unity already defined in (2.6).

3.2. *Boundary-layer approximation*

From (3.4), it can be seen that as $\varepsilon \to 0$, (3.1) is a singular eigenvalue problem. In the limit $\varepsilon = 0$, it yields $(\Sigma^2 + \Delta^2)\psi = 0$. If for some $r_0 > 0$, $\Sigma^2(r_0) = -\Delta^2(r_0)$, then the generalized solution $\psi(r) = \delta(r - r_0)$ satisfies the boundary value problem. Let us assume that $\Sigma^2(r_0) > 0$. Since by definition $\omega = \varepsilon^{-1}\Lambda(r_0) - \Delta(r_0)$, the generalized unstable eigensolution has complex frequency:

$$\omega_0 = \varepsilon^{-1}\Lambda_0 + \mathrm{i}\Sigma_0, \tag{3.5}$$

where $\Lambda_0 = \Lambda(r_0)$ and the 'growth rate' Σ_0 is the positive square root of $\Sigma^2(r_0)$ defined in (2.6). This is in agreement with the WKB results of Eckhoff & Storesletten given on the left-hand side of (2.12). But such a generalized solution is neither smooth nor square integrable and belongs to the continuous part of the spectrum. We shall now use a boundary-layer approximation to construct smooth discrete eigenmodes.

We expect Σ_0 to be an upper bound for the growth rate ω_I of such modes. From (2.4), this is ensured if $\Delta'(r_0) = 0$. Therefore in the rest of the paper, we assume that

$$\Lambda_0' = \tilde{m}\Omega_0' + \tilde{k}W_0' = 0, \tag{3.6}$$

where $\Lambda_0' = \Lambda'(r_0)$. This condition, consistent with (2.11), was deduced by Leibovich & Stewartson using different arguments.

In order to construct boundary-layer solutions of (3.1) when $\varepsilon \ll 1$, which are localized in the vicinity of r_0 where $\Sigma_0 > 0$ and $\Lambda_0' = 0$, we set

$$\tilde{r} = (r - r_0)/\varepsilon^{\alpha_1}, \quad \tilde{\omega} = (\omega - \omega_0)/\varepsilon^{\alpha_2}, \quad \tilde{\psi}(\tilde{r}) = \psi(r), \tag{3.7}$$

where $\alpha_{1,2}$ are positive scaling parameters to be determined, $\tilde{r}$ the inner spatial variable, and $\tilde{\omega}$ the yet unknown correction to the complex eigenvalue $\omega_0 = \varepsilon^{-1}\Lambda_0 + \mathrm{i}\Sigma_0$.

Expanding each term of $Q(r;\omega)$ in the vicinity of r_0 and using (3.7), the eigenvalue equation (3.1) may be written as $\mathrm{d}^2\tilde{\psi}/\mathrm{d}\tilde{r}^2+\tilde{q}(\tilde{r};\tilde{\omega})\tilde{\psi}=0$ with

$$\tilde{q}(\tilde{r};\tilde{\omega}) = 2\mathrm{i}S_0\tilde{\omega}\varepsilon^{(2\alpha_1+\alpha_2-2)} - \mathrm{i}S_0\Lambda_0''\tilde{r}^2\varepsilon^{(4\alpha_1-3)} + 2S_0\Sigma_0'\tilde{r}\varepsilon^{(3\alpha_1-2)} + 2\mathrm{i}S_0\Gamma_0\varepsilon^{(2\alpha_1-1)} + S_0\Sigma_0''\tilde{r}^2\varepsilon^{(4\alpha_1-2)} + \cdots. \quad (3.8)$$

Here $S_0=(\tilde{m}^2+r_0^2\tilde{k}^2)/(r_0^2\Sigma_0)$, and Γ_0 is a constant term defined later (§ 4).

By construction, the boundary-layer approximation makes sense only if $\tilde{q}(\tilde{r};\tilde{\omega})=O(1)$; furthermore, to remain an eigenvalue problem in terms of the inner variables $(\tilde{r}, \tilde{\omega}, \tilde{\psi})$, the resulting equation must include a dependence on $\tilde{\omega}$. This determines in a unique way the scaling parameters α_1 and α_2. Obviously from (3.8), various combinations are possible, depending on whether the coefficients Λ_0'', Σ_0', Σ_0'', etc., vanish or not. We first assume that $\Lambda_0''\neq 0$. Cases where Λ is uniform (which implies that $\Lambda'=\Lambda''=0$ everywhere) will be examined in § 4.

3.3. *The Leibovich–Stewartson modes*

In the general case, that is for arbitrary velocity profiles $\Omega(r)$ and $W(r)$, the condition (3.6) yields, for fixed $(\tilde{m},\tilde{k})$, the location r_0 of the mode. As we have assumed that $\Lambda_0''\neq 0$, the requirement that $\tilde{q}(\tilde{r};\tilde{\omega})=O(1)$ implies from (3.8) that

$$\alpha_1 = 3/4, \quad \alpha_2 = 1/2, \quad (3.9)$$

in agreement with Leibovich & Stewartson. The inner problem is then

$$\mathrm{d}^2\tilde{\psi}/\mathrm{d}\tilde{r}^2 + (2\mathrm{i}S_0\tilde{\omega} - \mathrm{i}S_0\Lambda_0''\tilde{r}^2)\tilde{\psi} = 0. \quad (3.10)$$

Inner boundary conditions are $\tilde{\psi}\to 0$ when $\tilde{r}\to\pm\infty$. Setting

$$\lambda = (4\mathrm{i}S_0\Lambda_0'')^{1/2}, \quad x = \tilde{r}\lambda^{1/2}, \quad E = 2\mathrm{i}S_0\tilde{\omega}/\lambda, \quad f(x) = \tilde{\psi}(\tilde{r}),$$

then (3.10) yields the following eigenvalue problem:

$$\mathrm{d}^2 f/\mathrm{d}x^2 + \left(E - \tfrac{1}{4}x^2\right) f = 0, \quad f(\pm\infty) = 0. \quad (3.11)$$

Non-trivial solutions that decay at infinity exist provided that the eigenvalues satisfy $E^{(n)}=\frac{1}{2}+n$ where $n=0, 1, 2, \ldots$. Eigenfunctions are $f_n(x)=\exp(-\frac{1}{4}x^2)\mathrm{He}_n(x)$, where $\mathrm{He}_0(x)=1$, $\mathrm{He}_1(x)=x$, $\mathrm{He}_2(x)=x^2-1, \ldots$ are Hermite polynomials.

Since $S_0>0$, then $\lambda=(1\pm\mathrm{i})(2S_0|\Lambda_0''|)^{1/2}$ where the '+' (resp. '−') is taken if $\Lambda_0''>0$ (resp. $\Lambda_0''<0$). The rescaled correction to the complex frequency is therefore

$$\tilde{\omega}^{(n)} = (\pm 1 - \mathrm{i})\left(\tfrac{1}{2}+n\right)\left(\tfrac{1}{2}|\Lambda_0''|/S_0\right)^{1/2}. \quad (3.12)$$

From (3.5), (3.7), (3.9) and (3.12), we conclude that the complex frequency of unstable modes has the following asymptotic discrete distribution when $\varepsilon \ll 1$:

$$\omega^{(n)} = \varepsilon^{-1}\Lambda_0 + \mathrm{i}\Sigma_0 + \varepsilon^{1/2}(\pm 1 - \mathrm{i})\left(\tfrac{1}{2}+n\right)\left(\tfrac{1}{2}|\Lambda_0''|/S_0\right)^{1/2} + \cdots. \quad (3.13)$$

If desired, higher-order corrections may be computed following a formal procedure described in Leibovich & Stewartson.

Thanks to (3.3) and (3.6), $\omega^{(n)}$ may be expressed as a function of m instead of ε. For instance, the leading term of the (real) frequency $\varepsilon^{-1}\Lambda_0$ is $m(\Omega_0-\Omega_0'W_0/W_0')$, whereas the growth rates reach, as $|m|\to\infty$, the asymptotic value Σ_0 given by the positive square root of $\Xi^2(r_0)$ defined in (2.9). Therefore, the unstable discrete spectrum is non-empty if $\Xi^2>0$ somewhere, an inequality equivalent to (1.1).

Eigenfunctions are complex Gaussian multiplied by Hermite polynomials. The compressible modes thus have a structure similar to the modes described by Leibovich & Stewartson in the incompressible case. Note that the above analysis is also valid for a non-homogeneous incompressible flow with radial density stratification if $C^2 = \infty$ is formally taken.

4. The case of uniform angular or axial velocity distributions

4.1. *Modes localized in the flow: Bayly's scaling*

In the previous section, we have constructed instability modes that are localized on the radius r_0 where $\Lambda_0' = 0$. However, if Λ is uniform in space then the construction of § 3.3 fails because the leading-order correction (3.12) to the complex frequency vanishes (since $\Lambda'' = 0$). Let us examine this case in detail. Λ is uniform if one of the following conditions is fulfilled: (i) both Ω and W are uniform; (ii) Ω is uniform and $k = 0$; (iii) W is uniform and $m = 0$.

Since $\Lambda' = 0$ everywhere in each of these cases, it is clear that the radius r_0 on which the eigenmode is localized cannot be determined by condition (3.6) as for the Leibovich–Stewartson modes. For fixed $(\tilde{m}, \tilde{k})$ however, there exists a non-zero radius r_0 on which $\Sigma(r)$ defined in (2.6) reaches its maximum value Σ_0. If the flow is unbounded, then this maximum is necessarily quadratic, that is such that $\Sigma_0' = 0$ and $\Sigma_0'' < 0$. If however the flow is bounded, the maximum may be reached inside the flow, or at a boundary. In the former case the maximum is quadratic, but in the latter case the maximum is generally such that $\Sigma_0' \neq 0$ on the boundary.

We first consider the case of quadratic maxima: we suppose that there exists $r_0 > 0$ inside the flow where $\Sigma_0 > 0$, $\Sigma_0' = 0$ and $\Sigma_0'' < 0$. Then (3.8) with $\Lambda_0'' = 0$ shows that the correct scaling of the inner variables (3.7) is

$$\alpha_1 = 1/2, \quad \alpha_2 = 1. \tag{4.1}$$

This scaling was discovered by Bayly (1988) for incompressible centrifugal instability.

In case (i) where both Ω and W are uniform, the inner problem is

$$\mathrm{d}^2\tilde{\psi}/\mathrm{d}\tilde{r}^2 + (2\mathrm{i}S_0(\tilde{\omega} + \Gamma_0) + S_0\Sigma_0''\tilde{r}^2)\tilde{\psi} = 0, \tag{4.2}$$

where $S_0 = (\tilde{m}^2 + r_0^2\tilde{k}^2)/(r_0^2\Sigma_0)$ and Γ_0 is a constant term defined as

$$\Gamma_0 = \frac{\tilde{m}\Omega}{S_0\Sigma_0}\left(\frac{2\tilde{k}^2}{\tilde{m}^2 + \tilde{k}^2 r_0^2} + \frac{2\Omega^2}{C_0^2} - \frac{R_0'}{r_0 R_0}\right). \tag{4.3}$$

The inner eigenvalue problem (4.2) may be put in the standard form (3.11) by the following change of variables:

$$\lambda = (-4S_0\Sigma_0'')^{1/2}, \quad x = \tilde{r}\lambda^{1/2}, \quad E = 2\mathrm{i}S_0(\tilde{\omega} + \Gamma_0)/\lambda, \quad f(x) = \tilde{\psi}(\tilde{r}).$$

We recall that the eigenvalues of (3.11) are $E^{(n)} = \frac{1}{2} + n$, and therefore

$$\tilde{\omega}^{(n)} = -\Gamma_0 - \mathrm{i}\left(\tfrac{1}{2} + n\right)(-\Sigma_0''/S_0)^{1/2}. \tag{4.4}$$

From (3.5), (3.7), (4.1) and (4.4), we conclude that the eigenvalues of unstable discrete modes have, when $\varepsilon \ll 1$, the following asymptotic behaviour:

$$\omega_R^{(n)} = \varepsilon^{-1}\Lambda_0 - \varepsilon\Gamma_0 + \cdots, \quad \omega_I^{(n)} = \Sigma_0 - \varepsilon\left(\tfrac{1}{2} + n\right)(-\Sigma_0''/S_0)^{1/2} + \cdots. \tag{4.5}$$

Here $\Lambda_0 = \tilde{m}\Omega + \tilde{k}W$. For fixed ε, the most amplified eigenmode corresponds to $n = 0$ and is a Gaussian centred on r_0.

From (2.3), (2.6) and (3.3), the asymptotic growth rate $\varSigma_0$ for large wavenumbers is the positive square root of $-N_0^2-(2\varOmega kr_0)^2/(m^2+k^2r_0^2)$, in agreement with (2.15) from the theory of Echkoff & Storesletten. Planar disturbances ($k=0$) are the most amplified and the flow is unstable if $N^2<0$ somewhere. Together with the results of § 2.1, we can conclude that the discrete spectrum of a swirling compressible flow with uniform angular and axial velocities is stable *if and only if* $N^2\geqslant 0$ everywhere. Since $N^2=-r\varOmega^2S'/\gamma$ in a perfect gas, then if $\varOmega$ and W are uniform, the flow is unstable if and only if its entropy $S(r)$ increases somewhere.

The accuracy of our asymptotic results may be checked thanks to a semi-circle theorem by Fung (1983). He considers the stability of an incompressible flow in solid-body rotation ($\varOmega$ uniform, $W=0$) with non-uniform density ($R'\neq 0$). With our notation, Fung proved that the complex frequency of two-dimensional disturbances ($k=0$) is such that

$$\left(\omega_R-\tfrac{1}{2}m\varOmega\right)^2+\omega_I^2=\left(\tfrac{1}{2}m\varOmega\right)^2. \tag{4.6}$$

Our results may be applied to this case if $C^2=\infty$. From (2.3), $N^2=r\varOmega^2R'/R$, so that the flow is unstable if and only if there exists some r_0 such that $\varSigma_0^2=-N_0^2>0$ is positive, that is if and only if $R_0'<0$. From (4.3) with $\tilde{k}=0$, we have $\varGamma_0=\tilde{m}\varSigma_0^2/\varOmega$, so that (4.5) yields in the present case, when $|m|\gg 1$

$$\omega_R^{(n)}=m\varOmega-\varSigma_0^2/(m\varOmega)+\cdots,\quad \omega_I^{(n)}=\varSigma_0-|m|^{-1}\left(\tfrac{1}{2}+n\right)\left(-r_0^2\varSigma_0\varSigma_0''\right)^{1/2}+\cdots.$$

Fung's equality (4.6) is satisfied with an error of order $|m|^{-1}$. Modes with similar structure have been described by Sipp *et al.* (2005) in the core of a vortex where the angular velocity is locally uniform.

We now turn to the case (ii) where $\varOmega$ is uniform but $W'\neq 0$. $\varLambda$ is uniform for planar disturbances ($\tilde{k}=0$). The modes have the same structure as those described by (4.5). In particular, they grow exponentially if $N^2<0$ somewhere, in agreement with (2.14). Together with the necessary condition (2.8), we can conclude that the flow is stable with respect to planar discrete eigenmodes *if and only if* $N^2\geqslant 0$ everywhere. However, since the axial velocity W is not uniform, nothing can be concluded for three-dimensional disturbances, except if there exists some r_0 on which $W_0'=0$. In that case, condition (3.6) is locally satisfied and the corresponding modes have a structure similar to those described by (3.13). However it may be shown that these three-dimensional modes are less amplified than planar modes.

We finally turn to case (iii) where $\varOmega'\neq 0$ and W is uniform. In that case $\varLambda$ is uniform for axisymmetric disturbances ($\tilde{m}=0$). This case corresponds to the compressible counterpart of Rayleigh's criterion for centrifugal instability. It has already been described in our previous work (Le Duc & Leblanc 1999): when $|k|\gg 1$, the complex frequencies of unstable eigenmodes behave as

$$\omega^{(n)}=kW+\mathrm{i}\varSigma_0-\mathrm{i}|k|^{-1}\left(\tfrac{1}{2}+n\right)(-\varSigma_0\varSigma_0'')^{1/2}+\cdots,$$

the flow being unstable if $\varSigma_0^2=-N_0^2-\varPhi_0>0$. Together with Howard's criterion (2.7), we can conclude that the flow is stable to axisymmetric disturbances *if and only if* $N^2+\varPhi\geqslant 0$ everywhere.

4.2. *Modes localized on a solid boundary: Reid's scaling*

In § 4.1, we have examined in detail the various cases for which the Leibovich–Stewartson scaling (3.9) does not hold, namely when $\varLambda$ is uniform. In these cases, we have seen that the instability modes are localized where the asymptotic growth rate $\varSigma(r)$ reaches its maximum $\varSigma_0$. The asymptotic structure of the eigenmodes has been constructed with the assumption that for fixed $(\tilde{m}, \tilde{k})$, a quadratic maximum is reached on r_0, that is $\Sigma_0' = 0$ and $\Sigma_0'' < 0$. Such a maximum can always be found in an unbounded flow. However, if the flow is contained inside cylindrical boundaries, the maximum of $\Sigma(r)$ may be reached on the inner or the outer boundaries so that $\Sigma_0' \neq 0$ there. Thus the analysis of §4.1 fails in this case.

We consider for simplicity a flow in solid-body rotation (Ω uniform and $W = 0$) perturbed by planar disturbances ($\tilde{k} = 0$). For solid-body rotation, we obtain from (2.3) and (2.6) $\Sigma^2 = -N^2$. Therefore if the entropy of the base flow increases monotonically, the maximum of $\Sigma(r)$ is reached on the outer boundary (say r_0) where $\Sigma_0' > 0$. (The case where $\Sigma_0' < 0$ on the inner boundary may be solved in a similar way.)

Recall that we still consider situations where Λ is uniform so that $\Lambda_0'' = 0$ in (3.8); the correct scaling of the inner variables (3.7) is now

$$\alpha_1 = \alpha_2 = 2/3.$$

This is the scaling first discovered by Reid (1960) for centrifugal instability of Couette flow in the narrow-gap approximation. With that scaling, the inner problem is

$$\mathrm{d}^2\tilde{\psi}/\mathrm{d}\tilde{r}^2 + (2\mathrm{i}S_0\tilde{\omega} + 2S_0\Sigma_0'\tilde{r})\tilde{\psi} = 0,$$

where $S_0 = \tilde{m}^2/(r_0^2\Sigma_0)$ here. In terms of the inner variables, the outer boundary corresponds to $\tilde{r} = 0$, the flow domain being such that $\tilde{r} < 0$. Therefore, boundary conditions are $\tilde{\psi} \to 0$ as $\tilde{r} \to -\infty$ and $\tilde{\psi} = 0$ on $\tilde{r} = 0$. Setting

$$\lambda = (2S_0\Sigma_0')^{2/3}, \quad x = -\tilde{r}\lambda^{1/2}, \quad E = 2\mathrm{i}S_0\tilde{\omega}/\lambda, \quad f(x) = \tilde{\psi}(\tilde{r}),$$

we obtain the following eigenvalue problem:

$$\mathrm{d}^2 f/\mathrm{d}x^2 + (E - x)f = 0, \quad f(0) = f(+\infty) = 0.$$

Solutions are Airy functions $f_n(x) = \mathrm{Ai}(x - E^{(n)})$ and the boundary condition at $x = 0$ gives the eigenvalue relation $\mathrm{Ai}(-E^{(n)}) = 0$. The zeros of the Airy function must be computed numerically. The first zeros are approximately $E^{(0)} \approx 2.33811$, $E^{(1)} \approx 4.08795$, $E^{(2)} \approx 5.52056, \ldots$. As a result, when $|m| \gg 1$, the complex frequency of the corresponding unstable modes behaves as

$$\omega^{(n)} = m\Omega + \mathrm{i}\Sigma_0 - \tfrac{1}{2}\mathrm{i}|m|^{-2/3}E^{(n)}\left(4r_0^2\Sigma_0\Sigma_0'^2\right)^{1/3} + \cdots.$$

Every other cases considered in §4.1 may be described in a similar way. For instance it may be shown that for centrifugal instability (W uniform, $m = 0$), the eigenvalues of those boundary modes are, when $|k| \gg 1$

$$\omega^{(n)} = kW + \mathrm{i}\Sigma_0 - \tfrac{1}{2}\mathrm{i}|k|^{-2/3}E^{(n)}\left(4\Sigma_0\Sigma_0'^2\right)^{1/3} + \cdots.$$

This result is consistent with numerical computations reported in Le Duc & Leblanc (1999) (see also Le Duc 2001). Indeed, we observed in such a case that, as the wavenumber k increases, the eigenmodes flatten near the boundary and their spatial scale is of order $|k|^{-2/3}$ (no explanation had been given). The same result has been obtained in a different manner by Billant & Gallaire (2005).

5. Summary

The asymptotic structure of the discrete eigenmodes associated with compressible swirling flows with velocity $V(r)\boldsymbol{e}_\theta + W(r)\boldsymbol{e}_z$, density $R(r)$, and speed of sound $C(r)$ has been described for disturbances with large wavenumbers. It has been shown that such disturbances are unstable when the local criterion of Eckhoff & Storesletten (1978) is satisfied: the discrete unstable spectrum is non-empty if (1.1) holds somewhere.

In the general case where V/r and W are not uniform, this provides a sufficient condition for instability. Compressible eigenmodes have a structure similar to the structure discovered by Leibovich & Stewartson (1983) in the incompressible case: they are centred on the points where $m(V/r)' + kW' = 0$ and their characteristic length scale is of order $|m|^{-3/4}$, where m and k are the azimuthal and axial wavenumbers.

Results differ for uniform angular or axial velocity distributions. If W is uniform, thanks to a criterion by Howard (1973), (1.1) becomes a *necessary and sufficient* condition for instability with respect to axisymmetric disturbances: this is Rayleigh's centrifugal instability criterion for compressible flows. If both W and $V/r = \Omega$ are uniform, thanks to work by Lalas (1975) and Warren (1975), then we can conclude that the discrete spectrum is unstable *if and only if* $R'/R - r\Omega^2/C^2 < 0$ for some $r > 0$. For a perfect gas, this shows that the flow is unstable if and only if the entropy increases somewhere. In a non-homogeneous incompressible rotating flow, instability arises if density decreases somewhere. Such a mechanism has been recently identified by Sipp *et al.* (2005). We have shown that instability modes have a different structure, depending on whether they are localized off or on the possible boundaries. They scale with $|m|^{-1/2}$ in the former case, and with $|m|^{-2/3}$ in the latter. For solid-body rotation, we also proved that the most unstable large-wavenumber disturbances are planar.

From a mathematical point of view, we have seen that the local theory yields the asymptotic value of the discrete eigenfrequencies, as $|m|$ and/or $|k| \to \infty$. Therefore, the points of the continuous spectrum described by the local theory are accumulation points for the discrete spectrum. One may wonder if this is generally true. The answer is at present not known.

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